

# The COM-negative binomial distribution: modeling overdispersion and ultrahigh zero-inflated count data

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## Abstract

In this paper, we focus on the COM-type negative binomial distribution with three parameters, which belongs to COM-type  $(a, b, 0)$  class distributions and family of equilibrium distributions of arbitrary birth-death process. Besides, we show abundant distributional properties such as overdispersion and underdispersion, log-concavity, log-convexity (infinite divisibility), pseudo compound Poisson, stochastic ordering and asymptotic approximation. Some characterizations including sum of equicorrelated geometrically distributed r.v.'s, conditional distribution, limit distribution of COM-negative hypergeometric distribution, and functional operator characterization are given for theoretical properties. COM-negative binomial distribution was applied to overdispersion and ultrahigh zero-inflated data sets. We employ maximum likelihood method to estimate the parameters and the goodness-of-fit are evaluated by the discrete Kolmogorov-Smirnov test.

*Keywords:* overdispersion, zero-inflated data, compound Poisson distribution, infinite divisibility, discrete Kolmogorov-Smirnov test

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## 1. Introduction

Before 2005, Conway-Maxwell-Poisson distribution (denoted as COM-Poisson distribution) had been rarely used since Conway and Maxwell (1962) briefly introduced it for modeling of queuing systems with state-dependent service time, see also Wimmer and Altmann (1999), Wimmer et al. (1995). About ten years ago, the COM-Poisson distribution with two parameters was revived by Shmueli et al. (2005) as a generalization of Poisson distribution. More recently, there has been a fast growth of researches on COM-Poisson distribution in terms of related statistical theory and applied methodology, see Sellers et al. (2012) and the references therein. The probability mass function(p.m.f.) is given by

$$P(X = k) = \frac{\lambda^k}{(k!)^\nu} \cdot \frac{1}{Z(\lambda, \nu)}, (k = 0, 1, 2, \dots), \quad (1)$$

where  $\lambda, \nu > 0$  and  $Z(\lambda, \nu) = \sum_{i=0}^{\infty} \frac{\lambda^i}{(i!)^\nu}$ . We denote (1) as  $X \sim \text{CMP}(\lambda, \nu)$ .

Kokonendji et al. (2008) proved that COM-Poisson distribution was overdispersed when  $\nu \in [0, 1)$  and underdispersed when  $\nu \in (1, +\infty)$ . Another extension of Poisson is negative binomial, which is a noted discrete distribution with overdispersion property and is widely applied in actuarial sciences(see Denuit et al. (2007), Kaas et al. (2008)). The p.m.f. of the negative binomial r.v  $X$  is

$$P(X = k) = \frac{\Gamma(r+k)}{k!\Gamma(r)} p^k (1-p)^r \quad \text{for } k = 0, 1, 2, \dots \quad (2)$$

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where  $r \in (0, \infty)$  and  $p \in (0, 1)$ .

In this paper, we introduce a new extension of the negative binomial distribution (denoted by CMNB) depending on three parameters by replacing  $\frac{\Gamma(r+k)}{k!\Gamma(r)}$  in (2) with  $(\frac{\Gamma(r+k)}{k!\Gamma(r)})^\nu$  and divide the normalization constant  $C(r, \nu, p) = \sum_{i=0}^{\infty} (\frac{\Gamma(r+i)}{i!\Gamma(r)})^\nu p^i (1-p)^r$ .

**Definition 1.1.** A r.v.  $X$  is said to follow COM-negative binomial distribution (CMNB( $r, \nu, p$ )) with three parameters ( $r, \nu, p$ ) if the p.m.f. is given by

$$P(X = k) = \frac{(\frac{\Gamma(r+k)}{k!\Gamma(r)})^\nu p^k (1-p)^r}{\sum_{i=0}^{\infty} (\frac{\Gamma(r+i)}{i!\Gamma(r)})^\nu p^i (1-p)^r} = \left( \frac{\Gamma(r+k)}{k!\Gamma(r)} \right)^\nu p^k (1-p)^r \frac{1}{C(r, \nu, p)}, \quad (k = 0, 1, 2, \dots), \quad (3)$$

where  $r, \nu \in (0, \infty)$  and  $p \in (0, 1)$ .

In another point of view, the alternative form of (3) can be written as

$$P(X = k) = \frac{[\frac{\Gamma(r+k)}{k!\Gamma(r)} \tilde{p}^k (1-\tilde{p})^r]^\nu}{\sum_{i=0}^{\infty} [\frac{\Gamma(r+i)}{i!\Gamma(r)} \tilde{p}^i (1-\tilde{p})^r]^\nu} = \left( \frac{\Gamma(r+k)}{k!\Gamma(r)} \right)^\nu \tilde{p}^k (1-\tilde{p})^r \frac{1}{C(r, \nu, \tilde{p})}, \quad (k = 0, 1, 2, \dots), \quad (4)$$

where  $\tilde{p} = p^{1/\nu}$ .

When  $r \leq 1$ , we will show that COM-negative binomial (3) is discrete compound Poisson, which has wide application in risk theory (includes non-life insurance) as well, see Zhang et al. (2014) and the references therein. We plot 12 cases of COM-negative binomial p.m.f. in Figure 1 in the end of this section.

It is easy to see that our COM-negative binomial distribution belongs to the COM-type extension of  $(a, b, 0)$  class. The  $(a, b, 0)$  class distribution is a famous family of distributions which sometimes refers to Katz class (see remarks in section 2.3.1 of Johnson et al. (2005)). It has significant applications in non-life insurance mathematics, especially for modelling claim counts( loss models, collective risk models), see Kaas et al. (2008), Denuit et al. (2007). A classic result in non-life insurance textbooks states that the  $(a, b, 0)$  class distribution only contains degenerate, binomial, Poisson and the negative binomial distribution. After adding a new parameter  $\nu \in \mathbb{R}^+$ , we define the COM-type extension of  $(a, b, 0)$  distribution, and it is convenient to see that degenerate, COM-Poisson, COM-binomial and COM-negative binomial belong to this class of distributions.

Shmueli et al. (2005) firstly proposed the COM-binomial distribution which is presented as a sum of equicorrelated Bernoulli variables. Borges et al. (2014) studied some properties and an asymptotic approximation (e.g. COM-binomial approximates to COM-Poisson under some conditions) of this family of distributions in detail. We will show that some results of COM-Poisson can be extended in our COM-negative binomial distribution. Kadane (2016) gives the exchangeably properties, sufficient statistics and multivariate extension of COM-binomial distribution.

Another variant of COM-negative binomial distribution has been studied by Imoto (2014), it just replaces the term  $\Gamma(r+k)$  in (2) by  $\Gamma(r+k)^\nu$  and then divides the normalization constant. We will give adequate reasons to support our extension in succeeding sections. Chakraborty and Imoto (2016) considered the extended COM-Poisson distribution (ECOMP( $r, \theta, \alpha, \beta$ )):

$$P(X = k) = \frac{\Gamma(r+k)^\beta}{(k!)^\alpha} \theta^k / \sum_{i=1}^{\infty} \frac{\Gamma(r+i)^\beta}{(i!)^\alpha} \theta^i \quad (k = 0, 1, 2, \dots), \quad (5)$$

where the parameter space is  $(r \geq 0, \theta > 0, \alpha > \beta) \cup (r > 0, 0 < \theta < 1, \alpha = \beta)$ . ECOMP distribution combines Imoto (2014)'s extension ( $\alpha = 1$ ) and our extension of COM-negative binomial distribution ( $\alpha = \beta = \nu$ ). The COM-Poisson is a special case of ECOMP when  $\beta = 0$ . ECOMP distribution has the queuing systems characterization (birth-death process with arrival rate  $\lambda_k = (r+k)^\nu$  and service rate  $\mu_k = k^\nu \mu$  for  $k \geq 1$ ), see also Brown and Xia (2001) for arbitrary birth-death process characterization.

The rest of the article is organized as the follows. In section 2, we propose the COM-type  $(a, b, 0)$  class distributions, and demonstrate some example of COM-type  $(a, b, 0)$  class which includes the COM-negative binomial. Further more, some properties of COM-negative binomial are given: Renyi entropy and Tsallis entropy representation, overdispersion and underdispersion, log-concavity, log-convexity (infinite divisibility), pseudo compound Poisson, stochastic ordering and asymptotic approximation. In section 3, some conditional distribution characterizations and Stein identity characterization are presented by using related lemmas, and we also show that COM-negative hypergeometric can approximate to COM-negative binomial. In section 4, inverse method were introduced to generate COM-negative binomial distributed random variables. Section 5 then estimates the parameters by maximum likelihood method, in which the initial values are provided by recursive formula. In section 6, two simulated data sets and two applications to actuarial claim data sets are given as examples. In section 7, we provide some potential and further research suggestions based on the properties and characterizations of COM-negative binomial distribution.

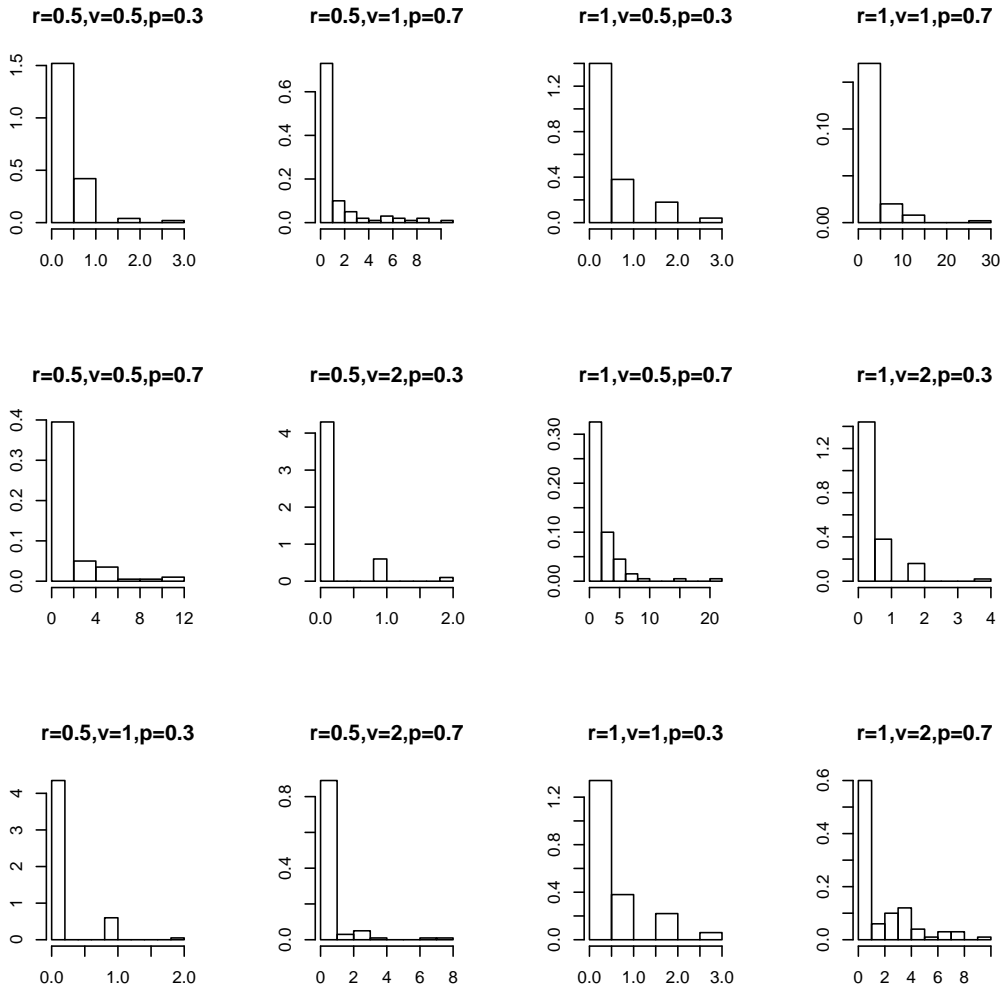


Figure 1: Some plots of p.m.f. (3) for  $r \in \{0.5, 1\}$ ,  $v \in \{0.5, 1, 2\}$  and  $p \in \{0.3, 0.7\}$ .

## 2. Properties

### 2.1. Recursive formula and ultrahigh zero-inflated property

The recursive formula (or ratios of consecutive probabilities) is given by

$$\frac{P(X = k)}{P(X = k - 1)} = p \left( \frac{\Gamma(k + r)}{k! \Gamma(r)} \right)^\nu / \left( \frac{\Gamma(k - 1 + r)}{(k - 1)! \Gamma(r)} \right)^\nu = p \cdot \left( \frac{k - 1 + r}{k} \right)^\nu, \quad (6)$$

We say that a zero-inflated count data  $X$  following some discrete distribution is ultrahigh zero-inflated if  $P(X = 0)/P(X = 1) \gg 1$ . For COM-negative binomial case with two parameters, we have  $\frac{P_{\text{CMNB}}(X=0)}{P_{\text{CMNB}}(X=1)} = \frac{1}{p} \left( \frac{1}{r} \right)^\nu$ ; and for negative binomial case, we get  $\frac{P_{\text{NB}}(X=0)}{P_{\text{NB}}(X=1)} = \frac{1}{pr}$ . If we choose  $r < 1$  and  $\nu > 1$  in COM-negative binomial case, then this three-parameter case is more flexible to deal with the ratio  $\frac{P(X=0)}{P(X=1)}$  comparing to the two-parameter case, as  $\frac{1}{p} \left( \frac{1}{r} \right)^\nu \gg \frac{1}{pr}$  when  $\nu > 1$ . (For examples, the plots of COM-negative binomial distribution with  $\nu = 2$  in Figure 1;  $\nu = 10.4$  in Table 4 of section 6.2). In the insurance company, the more zero insurance claims, the less risk to bankrupt.

The following definition provides a generalization of  $(a, b, 0)$  class distribution.

**Definition 2.1.** Let  $X$  be a discrete r.v., if  $p_k = P(X = k)$  satisfies the recursive formula

$$p_k = \left( a + \frac{b}{k} \right)^\nu p_{k-1}, \quad (k = 1, 2, \dots) \quad (7)$$

for some constants  $a, b \in \mathbb{R}$  and  $\nu \in \mathbb{R}^+$ , then we call it COM-type  $(a, b, 0)$  class distribution. We denote this class as  $\text{COM}(a, b, \nu, 0)$ .

The COM-Poisson distribution  $\text{CMP}(\lambda, \nu)$  satisfies the case  $a = 0$ , since  $p_k = \frac{\lambda}{k} p_{k-1}$  and  $b = \lambda^{1/\nu}$ .

From (6), it is easy to see that COM-negative binomial distribution belongs to COM-type  $(a, b, 0)$  class distribution with  $p_0 = 1/C(r, \nu, p)$ .

$$\frac{p_k}{p_{k-1}} = p \left( 1 + \frac{r-1}{k} \right)^\nu = \left( p^{1/\nu} + \frac{(r-1)p^{1/\nu}}{k} \right)^\nu. \quad (8)$$

The COM-binomial distribution (CMB), see Shmueli et al. (2005), Borges et al. (2014), with p.m.f.

$$P(X = k) = \frac{\binom{m}{k}^\nu p^k (1-p)^{m-k}}{\sum_{i=0}^m \binom{m}{i}^\nu p^i (1-p)^{m-i}}, \quad k = 0, 1, \dots, m, \quad (9)$$

where  $\nu \in \mathbb{R}^+, m \in \mathbb{Z}^+, p \in (0, 1)$ . We denote (9) as  $X \sim \text{CMB}(m, p, \nu)$ .

Since the ratio of consecutive probabilities is  $\frac{P(X=k)}{P(X=k-1)} = \frac{p}{1-p} \left( \frac{m+1-k}{k} \right)^\nu$ , COM-binomial distribution belongs to COM type  $(a, b, 0)$  class distribution.

**Remark 1:** As we know, the  $(a, b, 0)$  class distribution only contains degenerate distribution, binomial distribution, Poisson distribution and the negative binomial distribution. But there are other distributions belongs to COM type  $(a, b, 0)$  class. For example,  $m \in \mathbb{Z}^+$  can be replaced by  $m \in \mathbb{R}^+$  in (9), and the p.m.f. is given by

$$P(X = k) = \frac{\binom{m}{k}^2 p^k (1-p)^{m-k}}{\sum_{i=0}^{\infty} \binom{m}{i}^2 p^i (1-p)^{m-i}}, \quad k = 0, 1, 2, \dots,$$

where  $p/(1-p) < 1$  such that  $\sum_{i=0}^{\infty} \binom{m}{i}^2 p^i (1-p)^{m-i} < \infty$ .

**Remark 2:** Brown and Xia (2001) considered the very large class of stationary distribution of birth-death process with arrival rate  $\lambda_k$  and service rate  $\mu_k$  by the recursive formula:

$$\frac{P(X = k)}{P(X = k - 1)} = \frac{\lambda_{k-1}}{\mu_k}, \quad (k = 1, 2, \dots).$$

Thus we can construct a birth-death process with arrival rate  $\lambda_k = c[a(k + 1) + b]^\nu, (k = 1, 2, \dots)$  and service rate  $\mu_k = ck^\nu, (k = 1, 2, \dots; c \text{ is a positive constant})$ , which characterizes the COM-type  $(a, b, 0)$  class distribution.

### 2.2. Related to Rényi entropy and Tsallis entropy

Notice the Rényi entropy(see Rényi (1961)) in the information theory, which generalizes the Shannon entropy. The Renyi entropy of order  $\alpha$  of a discrete r.v.  $X$ :

$$H_\alpha^R(X) = \frac{1}{1 - \alpha} \ln \sum_{i=0}^{\infty} [P(X = i)]^\alpha, (\alpha \neq 1).$$

Let  $X$  be negative binomial distributed in (2) and  $X_\nu$  be COM-negative binomial distributed in (4). Then the normalization constant  $C(r, \nu, \tilde{p})$  in (4) has Rényi entropy representation  $C(r, \nu, \tilde{p}) = e^{(1-\nu)H_\nu^R(X)}$ , so  $P(X_\nu = x) = P^\nu(X = x)/e^{(1-\nu)H_\nu^R(X)}$ .

Another generalization of Shannon entropy in physic is the Tsallis entropy. For r.v.  $X$ , its Tsallis entropy of order  $\alpha$  is defined by

$$H_\alpha^T(X) = \frac{1}{1 - \alpha} \left( \sum_{i=0}^{\infty} [P(X = i)]^\alpha - 1 \right), (\alpha \neq 1).$$

This entropy was introduced by Tsallis (1988) as a basis for generalizing the Boltzmann-Gibbs statistics.

Also, the normalization constant  $C(r, \nu, \tilde{p})$  in (4) has Tsallis entropy representation  $C(r, \nu, \tilde{p}) = 1 + (1 - \nu)H_\nu^T(X)$ , then  $P(X_\nu = x) = P^\nu(X = x)/[1 + (1 - \nu)H_\nu^T(X)]$ .

### 2.3. Log-concave, Log-convex, Infinite divisibility

This subsection deals with log-concavity and log-convexity of the COM-negative binomial distribution. A discrete distribution with  $p_k = P(X = k)$  is said to have log-concave (log-convex) p.m.f. if

$$\frac{p_{k+1}p_{k-1}}{p_k^2} = \frac{p_{k+1}}{p_k} / \frac{p_k}{p_{k-1}} \leq (\geq) 1, \quad k \geq 1.$$

**Lemma 2.1.** *The COM-negative binomial distribution is log-concave if  $r \geq 1$  and log-convex if  $r \leq 1$ .*

*Proof.* In fact, using the ratio of consecutive probabilities (6), we have

$$M = \frac{p_{k+1}}{p_k} / \frac{p_k}{p_{k-1}} = p \left( \frac{r+k}{k+1} \right)^\nu / p \left( \frac{r+k-1}{k} \right)^\nu = \left( \frac{k^2 + kr}{k^2 + kr + r - 1} \right)^\nu.$$

Then  $M \leq 1$  iff  $r \geq 1$ (log-concave) and  $M \geq 1$  iff  $r \leq 1$ (log-convex). □

**Remark 3:** Ibragimov (1956) called a distribution strongly unimodal if it is unimodal and its convolution with any unimodal distribution is unimodal. He showed that the strongly unimodal distributions is equal to the log-concave distributions. So COM-negative binomial distribution has strong unimodality(see Figure 1 for example) when  $r \geq 1$ .

Steutel (1970) showed that all log-convex discrete distributions are infinitely divisible, the background and detailed proof can be found in Steutel and van Harn (2003). Then we obtain infinite divisibility of COM-negative binomial distribution when  $r \leq 1$ .

**Corollary 2.1.** *The COM-negative binomial distribution (3) is discrete infinitely divisible (discrete compound Poisson distribution) if  $r \leq 1$ .*

Feller's characterization of the discrete infinite divisibility showed that a non-negative integer valued r.v.  $X$  is infinitely divisible if and only if its distribution is a discrete compound Poisson distribution with p.g.f.:

$$G(z) = \sum_{k=0}^{\infty} p_k z^k = e^{\sum_{i=1}^{\infty} \alpha_i \lambda (z^i - 1)}, (|z| \leq 1) \quad (10)$$

where  $\sum_{i=1}^{\infty} \alpha_i = 1, \alpha_i \geq 0, \lambda > 0$ .

For a theoretical treatment of discrete infinite divisibility (or discrete compound Poisson distribution), we refer readers to section 2 of Steutel and van Harn (2003), section 9.3 of Johnson et al. (2005), Zhang and Li (2016).

Considering some  $\alpha_i$  being negative in (10), it turns into a generalization of the discrete compound Poisson distribution:

**Definition 2.2** (Discrete pseudo compound Poisson distribution). *If a discrete r.v.  $X$  with  $P(X = k) = p_k, k \in \mathbb{N}$ , has a p.g.f. of the form*

$$G(z) = \sum_{k=0}^{\infty} p_k z^k = \exp \left\{ \sum_{i=1}^{\infty} \alpha_i \lambda (z^i - 1) \right\}, \quad (11)$$

where  $\sum_{i=1}^{\infty} \alpha_i = 1, \sum_{i=1}^{\infty} |\alpha_i| < \infty, \alpha_i \in \mathbb{R}$ , and  $\lambda > 0$ , then  $X$  is said to follow a discrete pseudo compound Poisson distribution, abbreviated as DPCP.

Next, we will give two lemmas on the non-vanishing p.g.f. characterization of DPCP, see Zhang et al. (2014) and Zhang et al. (2017).

**Lemma 2.2.** *Let  $p_k = P(X = k)$ , for any discrete r.v.  $X$ , its p.g.f.  $G(z) = \sum_{k=0}^{\infty} p_k z^k$  has no zeros in  $-1 \leq z \leq 1$  if and only if  $X$  is DPCP distributed.*

The proof of Lemma 2.2 is based on Wiener-Lévy theorem, which is a sophisticated theorem in Fourier analysis, see Zygmund (2002).

**Lemma 2.3.** (Lévy-Wiener theorem) *Let  $F(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}, \theta \in [0, 2\pi]$  be a absolutely convergent Fourier series with  $\|F\| = \sum_{k=-\infty}^{\infty} |c_k| < \infty$ . The value of  $F(\theta)$  lies on a curve  $C$ , and  $H(t)$  is an analytic (not necessarily single-valued) function of a complex variable which is regular at every point of  $C$ . Then  $H[F(\theta)]$  has an absolutely convergent Fourier series.*

**Lemma 2.4.** *For any discrete r.v.  $X$  with p.g.f.  $G(z) = \sum_{k=0}^{\infty} p_k z^k, (|z| \leq 1)$ . If  $p_0 > p_1 > p_2 > \dots$ , then  $X$  is DPCP distributed.*

*Proof.* First, we show that  $G(z)$  has no zeros in  $|z| < 1$ , since

$$\begin{aligned} |(1-z)G(z)| &= |p_0 - (p_0 - p_1)z - (p_1 - p_2)z^2 + \dots| \\ &\geq p_0 - |(p_0 - p_1)|z| + (p_1 - p_2)|z^2| + \dots \\ &> p_0 - |(p_0 - p_1) + (p_1 - p_2) + \dots| = p_0 - |p_0| = 0 \end{aligned}$$

And notice that  $G(1) = 1, G(-1) = p_0 - p_1 + p_2 - p_3 + \dots > 0$ , so  $z = \pm 1$  are not zeros point.  $\square$

The condition in the next corollary is weaker than that of Corollary 2.1, and the result (DPCP) is also weaker than Corollary 2.1 (DCP).

**Corollary 2.2.** *The COM-negative binomial distribution (3) is discrete pseudo compound Poisson distribution if  $(pr^\nu < 1, r > 1)$  or  $(r \leq 1)$ .*

*Proof.* On the one hand,  $r \leq 1$  deduces that COM-negative binomial belongs to discrete compound Poisson by Corollary 2.1, hence COM-negative binomial is discrete pseudo compound Poisson. On the other hand, by using **Lemma 2.4**, we need to guarantee that  $\frac{P(X=k)}{P(X=k-1)} = p(\frac{k-1+r}{k})^\nu < 1$  for  $k = 1, 2, \dots$ .  $p(\frac{k-1+r}{k})^\nu$  is a decreasing function with respect to  $k$  when  $r > 1$ ,  $\frac{P(X=k)}{P(X=k-1)}$  reaches its maximum  $pr^\nu$  as  $k = 1$ . So  $pr^\nu < 1$  is the other case.  $\square$

Applying the recurrence relation (Lévy-Adelson-Panjer recursion) of p.m.f. of DPCP distribution, see Remark 1 in Zhang et al. (2014)

$$P_{n+1} = \frac{\lambda}{n+1} [\alpha_1 P_n + 2\alpha_2 P_{n-1} + \dots + (n+1)\alpha_{n+1} P_0], \quad (P_0 = e^{-\lambda}, n = 0, 1, \dots)$$

and  $P_k = P(X = k) = (\frac{\Gamma(r+k)}{k!\Gamma(r)})^\nu p^k P_0$ , then the DPCP parametrization  $(\lambda, \alpha_1, \alpha_2, \dots)$  of COM-negative binomial distribution is determined by the following system of equations:

$$(\frac{\Gamma(r+n+1)}{(n+1)!})^\nu p^{n+1} = \frac{\lambda}{n+1} [\alpha_1 (\frac{\Gamma(r+n)}{n!})^\nu p^n + 2\alpha_2 (\frac{\Gamma(r+n-1)}{(n-1)!})^\nu p^{n-1} + \dots + (n+1)\alpha_{n+1}], \quad (n = 0, 1, \dots),$$

where  $\lambda = \log P_0$ .

#### 2.4. Overdispersion and underdispersion

In statistics, for a given random sample  $X$ , overdispersion means that  $EX < \text{Var}X$ . Conversely, underdispersion means that  $EX > \text{Var}X$ . Moreover, equal-dispersion means that  $EX = \text{Var}X$ . Gmez-Dniz (2011) summerized the phenomena of insurance count claims data, which were characterized by two features: (i) Overdispersion, i.e., the variance is greater than the mean; (ii) Zero-inflated, i.e. the presence of a high percentage of zero values in the empirical distribution.

The COM-negative binomial distribution belongs to the family of weighted Poisson distribution (see Kokonendji et al. (2008)) with p.m.f.

$$P(X = k) = \frac{w(k)}{E[w(X)]} \cdot \frac{\theta^k}{k!} e^{-\theta}, \quad (12)$$

where  $w(k)$  is a non-negative weighted function.

Then weighted Poisson representation of COM-negative binomial distribution is

$$P(X = k) = \frac{(1-p)^r e^p}{C(r, \nu, p)} [\Gamma(1+k)]^{1-\nu} [\Gamma(r+k)]^\nu \frac{p^k}{k!} e^{-p}, \quad (k = 0, 1, 2, \dots).$$

Therefore, COM-negative binomial distribution in (3) can be seen as a weighted Poisson distribution with weighted function

$$f(k, r, \nu) = w(k) = [\Gamma(1+k)]^{1-\nu} [\Gamma(r+k)]^\nu. \quad (13)$$

Theorem 3 and its corollary in Kokonendji et al. (2008) provide an “iff” condition to prove overdispersion and underdispersion of the weighted Poisson distribution.

**Lemma 2.5.** *Let  $X$  be a weighted Poisson random variable with mean  $\theta > 0$ , and let  $w(k), k \in \mathbb{N}$  be a weighted function not depending on  $\theta$ . Then, weighted function  $k \mapsto w(k)$  is logconvex (logconcave) iff the weighted version  $X_w$  of  $X$  is overdispersed (underdispersed).*

Kokonendji et al. (2008) applied it to show that COM-Poisson distribution is overdispersion if  $\nu < 1$  and underdispersion if  $\nu > 1$ . We employ their methods to get a criterion for overdispersion or underdispersion of COM-negative binomial distribution.

**Theorem 2.1.** Set  $\Delta_k = \sum_{i=0}^{\infty} (\frac{1-\nu}{(i+k+1)^2} + \frac{\nu}{(i+k+r)^2})$ . The COM-negative binomial distribution (3) is overdispersion if  $\Delta_k > 0$  ( $\forall k \in \mathbb{N}$ ) and underdispersion if  $\Delta_k < 0$  ( $\forall k \in \mathbb{N}$ ).

*Proof.* Function  $f(x)$  is logconvex(logconcave) if  $\frac{d^2 \log f(x)}{dx^2} > 0 (< 0)$ . Followed by the formula of logarithmic second derivative of Gamma function(see p54 of Temme (2011)),  $\frac{d^2 \log \Gamma(x)}{dx^2} = \sum_{i=0}^{\infty} \frac{1}{(x+i)^2}$ , we have

$$\frac{d^2 \log f(k, r, \nu)}{dk^2} = \frac{(1-\nu)d^2 \log \Gamma(k+1)}{dk^2} + \frac{\nu d^2 \log \Gamma(k+r)}{dk^2} = \sum_{i=0}^{\infty} (\frac{1-\nu}{(i+k+1)^2} + \frac{\nu}{(i+k+r)^2}), (\forall k \in \mathbb{N}).$$

Applying **Lemma 2.5**, the proof is complete.  $\square$

Then, the results of overdispersion can be easily obtained by **Theorem 2.1**.

**Corollary 2.3.** In these two cases: 1.  $\nu > 0, r < 1$ ; 2.  $\nu < 1, r \in \mathbb{R}^+$ . COM-negative binomial distribution is overdispersion.

**Remark 4:** The result of case 2 ( $\nu < 1, r \in \mathbb{R}^+$ ) can be also obtained from **Corollary 2.1** and overdispersion of discrete compound Poisson distribution (equivalently, the discrete infinitely divisible).

## 2.5. Stochastic ordering

Stochastic ordering is the concept of one r.v.  $X$  neither stochastically greater than, less than nor equal to another r.v.  $Y$ . There are plenty types of stochastic orders, which have various applications in risk theory. Firstly, we present 4 different definitions for discrete r.v.: usual stochastic order, likelihood ratio order, hazard rate order and mean residual life order.

1.  $X$  is stochastically less than  $Y$  in usual stochastic order (denoted by  $X \leq_{st} Y$ ) if  $S_X(n) \geq S_Y(n)$  for all  $n$ , where  $S_X(n) = P(X \geq n) = \sum_{k=n}^{\infty} p_k$  is the survival function  $X$  of with p.m.f.  $p_k$ .

2.  $X$  is stochastically less than  $Y$  in likelihood ratio order(denoted by  $X \leq_{lr} Y$ ) if  $\frac{g(n)}{f(n)}$  increases in  $n$  over the union of the supports of  $X$  and  $Y$ , where  $f(n)$  and  $g(n)$  denotes the p.m.f. of  $X$  and  $Y$ , respectively.

3.  $X$  is stochastically less than  $Y$  in hazard rate order(denoted by  $X \leq_{hr} Y$ ) if  $r_X(n) \geq r_Y(n)$  for all  $n$ , where the hazard function of a discrete r.v.  $X$  with p.m.f.  $p_k$  is defined as  $r_X(n) = p_n / \sum_{k=n}^{\infty} p_k$

4.  $X$  is stochastically less than  $Y$  in mean residual life order(denoted by  $X \leq_{MLR} Y$ ) if  $\mu_X(n) \geq \mu_Y(n)$  for all  $n$ , where the mean residual life function of a discrete r.v.  $X$  with p.m.f.  $p_k$  is defined as  $\mu_X(n) = E(X - n | X \geq n) = \sum_{k=n}^{\infty} k p_k / \sum_{k=n}^{\infty} p_k - n$ .

The relationship among the above four stochastic ordering are  $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{MLR} Y$  (see Theorem 1.C.1 of Shaked and Shanthikumar (2007)) and  $X \leq_{hr} Y \Rightarrow X \leq_{st} Y$  (see Theorem 1.B.1 of Shaked and Shanthikumar (2007)).

Gupta et al. (2014) gave the stochastic ordering between COM-Poisson r.v.  $X$  and Poisson distributed r.v.  $Y$  with same parameter  $\lambda$  in (1), that is  $X \leq_{lr} Y$ , therefore  $X \leq_{st} Y$ ,  $X \leq_{hr} Y$  and  $X \leq_{MLR} Y$ . In the following result we will show that COM-negative binomial distribution also has some stochastic ordering properties.

**Theorem 2.2.** Let  $X$  and  $Y$  be two r.v.'s following COM-negative binomial distribution with parameters  $(r, \nu_1, p)$  and  $(r, \nu_2, p)$ , respectively. If  $\nu_1 \leq \nu_2, r \geq 1$ , then  $X \leq_{lr} Y$ , hence  $X \leq_{st} Y$ ,  $X \leq_{hr} Y$  and  $X \leq_{MLR} Y$ .



*Proof.* Note that  $r \geq 1$ , we have  $\frac{\Gamma(r+n+1)}{(n+1)!} / \frac{\Gamma(r+n)}{n!} = \frac{r+n}{n+1} \leq 1$ . Then

$$\frac{P(Y = n)}{P(X = n)} = \left( \frac{\Gamma(r+n)}{n! \Gamma(r)} \right)^{\nu_2 - \nu_1} \frac{C(r, \nu_1, p)}{C(r, \nu_2, p)}, \quad (n = 0, 1, 2, \dots).$$

which is increasing in  $n$  as  $\nu_1 \geq \nu_2$ .  $\square$

Especially, assume that  $r$  is a positive integer, the COM-negative binomial should be called the COM-Pascal distribution. Let  $X$  be COM-negative binomial distributed and  $Y$  be negative binomial distributed with the same parameters  $r, p$ , it yields to  $X \leq_{lr} Y$  when  $\nu_2 > \nu_1 = 1$ .

The next theorem is proved in the view of weighted Poisson distribution (12) from weighted function of COM-negative binomial distribution. Example 1.C.59 of Shaked and Shanthikumar (2007) states the obvious lemma below:

**Lemma 2.6.** Define  $X_w$  as the r.v. with weighted density function  $f_w(x) = \frac{w(x)}{E[w(X)]} f(x)$ , ( $x \geq 0$ ), Similarly, for another nonnegative r.v.  $Y$  with density function  $g$ , define  $Y_w$  as the r.v. with the weighted density function  $g_w(y) = \frac{w(y)}{E[w(Y)]} g(y)$ , ( $y \geq 0$ ). If  $w(x)$  is an increasing function, then  $X \leq_{hr} Y \Rightarrow X_w \leq_{hr} Y_w$ .

**Theorem 2.3.** Let  $X$  and  $Y$  be two COM-negative binomial distributed with parameters  $(r, \nu, p_1)$  and  $(r, \nu, p_2)$ , respectively. If  $(p_1 \leq p_2, \nu \leq 1)$  or  $(p_1 \leq p_2, r \geq 1)$ , then  $X \leq_{lr} Y$ , and therefore  $X \leq_{st} Y$ ,  $X \leq_{hr} Y$  and  $X \leq_{MLR} Y$ .

*Proof.* For Poisson distributed  $X, Y$  with mean  $p_1, p_2$ , if  $p_1 \leq p_2$ , then  $P(Y = n)/P(X = n) = \left(\frac{p_2}{p_1}\right)^n e^{-(p_2 - p_1)}$  is increasing for all  $n$ . So  $X \leq_{hr} Y$ . From section 2.4, we know that COM-negative binomial is weight Poisson with weight (13).

On the one hand, when  $\nu \leq 1$ , we notice that weighted density function  $w(x) = [\Gamma(1+x)]^{1-\nu} [\Gamma(r+x)]^\nu$  for COM-negative binomial distribution is increasing with respect to  $x$ . On the other hand,  $w(x) = \Gamma(1+x) \left(\frac{\Gamma(r+x)}{\Gamma(1+x)}\right)^\nu$  is an increasing function with respect to  $x$  as  $r \geq 1$ , that is,  $X_w \leq_{hr} Y_w$ .  $\square$

## 2.6. Approximate to COM-Poisson distribution

The next theorem enable COM-negative binomial distribution to be a suitable generalization since its limit distribution is the COM-Poisson under some conditions. We prove that COM-negative binomial distribution converges to the COM-Poisson distribution when  $r$  goes to infinity.

**Theorem 2.4.** Suppose that r.v.  $X$  has COM-negative binomial distribution with parameters  $(r, \nu, p)$ , denote the p.m.f. as  $P(X = k | r, \nu, p)$ , and let  $\lambda = r^\nu \frac{p}{1-p}$ . Then

$$\lim_{r \rightarrow \infty} P(X = k | r, \nu, p) = \frac{\lambda^k}{(k!)^\nu} \cdot \frac{1}{Z(\lambda, \nu)}, \quad (k = 0, 1, 2, \dots). \quad (14)$$

*Proof.* Notice that  $p = \frac{\lambda}{r^\nu + \lambda}$ , substitute to p.m.f (3), then we obtain

$$P(X = k | r, \nu, p) = \frac{\lambda^k}{(k!)^\nu} \cdot \left( \frac{\Gamma(r+k)}{\Gamma(r)} r^k \right)^\nu \cdot \frac{1}{(1 + \lambda/r^\nu)^k} \bigg/ \frac{C(r, \nu, p)}{\left(\frac{r^\nu}{r^\nu + \lambda}\right)^r}.$$

Hence,

$$\lim_{r \rightarrow \infty} P(X = k | r, \nu, p) = \frac{\lambda^k}{(k!)^\nu} \bigg/ \frac{C(r, \nu, p)}{\left(\frac{r^\nu}{r^\nu + \lambda}\right)^r} = \frac{\lambda^k}{(k!)^\nu} \cdot \frac{1}{Z(\lambda, \nu)}$$

holds as  $\lim_{r \rightarrow +\infty} \left( \frac{\Gamma(r+k)}{\Gamma(r)} r^k \right)^\nu \cdot \frac{1}{(1 + \lambda/r^\nu)^k} = 1$ , ( $k = 0, 1, 2, \dots$ ) and

$$\lim_{r \rightarrow +\infty} \frac{C(r, \nu, p)}{\left(\frac{r^\nu}{r^\nu + \lambda}\right)^r} = \lim_{n \rightarrow +\infty} \lim_{r \rightarrow +\infty} \sum_{i=0}^n \frac{\lambda^i}{(i!)^\nu} \cdot \left( \frac{\Gamma(r+i)}{\Gamma(r)} r^i \right)^\nu \cdot \frac{1}{(1 + \lambda/r^\nu)^i} = \lim_{n \rightarrow +\infty} \sum_{i=0}^n \frac{\lambda^i}{(i!)^\nu} = Z(\lambda, \nu).$$

$\square$

### 3. Characterizations

#### 3.1. Sum of equicorrelated geometrically distributed r.v.

It is well known that the binomial r.v. can be seen as the sum of  $m$  independent Bernoulli r.v.  $Z_i$ .

$$S = Z_1 + Z_2 + \cdots + Z_m$$

where

$$P(Z_i = 1) = p, P(Z_i = 0) = 1 - p, i = 1, 2, \dots, m$$

$$P(S = k) = \binom{m}{k} p^k (1 - p)^{m-k}.$$

Shmueli et al. (2005) and Borges et al. (2014) mentioned that the COM-binomial distribution (9) can be presented as a sum of equicorrelated Bernoulli r.v.'s  $\{Z_i\}_{i=1}^m$  with joint distribution

$$P(Z_1 = z_1, \dots, Z_m = z_m) = \frac{\binom{m}{k}^{v-1} p^k (1 - p)^{m-k}}{\sum_{x_1=0}^1 \cdots \sum_{x_m=0}^1 \left( \sum_{i=1}^m x_i \right)^{v-1} p^{\sum_{i=1}^m x_i} (1 - p)^{m - \sum_{i=1}^m x_i}}, z = (z_1, \dots, z_m) \in \{0, 1\}^m,$$

where  $k = \sum_{i=1}^m z_i$ .

As we know, negative binomial distribution can be treated as the sum of  $m$  independent geometric r.v.'s  $Z_i$  ( $i = 1, \dots, m$ ):

$$S = Z_1 + Z_2 + \cdots + Z_m$$

$$P(S = x) = \binom{m+x-1}{x} p^x (1 - p)^m,$$

where  $P(Z_i = z_i) = p^{z_i} (1 - p)$ , ( $z_i = 1, 2, \dots$ ).

It is similar to see that the COM-negative binomial distribution can be interpreted as a sum of equicorrelated geometric r.v.'s  $Z_i$  ( $i = 1, \dots, m$ ) with joint distribution

$$P(Z_1 = z_1, \dots, Z_m = z_m) \propto \binom{m+x-1}{x}^{v-1} p^x (1 - p)^m, \quad (15)$$

where  $x = \sum_{i=1}^m z_i$ .

The reason is that we assume  $\{Z_i\}_{i=1}^m$  is equicorrelated, and  $x = \sum_{i=1}^m z_i$  has  $\binom{m+x-1}{x}$  feasible positive integer solutions, and each solution has probability  $P(Z_1 = z_1, \dots, Z_m = z_m)$ . Then, the  $\binom{m+x-1}{x}$  possible values of random vector  $(Z_1, \dots, Z_m)$  such that  $S = \sum_{i=1}^m Z_i$  is COM-negative binomial distributed, namely

$$\binom{m+x-1}{x} P(Z_1 = z_1, \dots, Z_m = z_m) \propto \binom{m+x-1}{x}^v p^x (1 - p)^m,$$

Thus we have (15).

#### 3.2. Conditional distribution

In this subsection, two conditional distribution characterizations are obtained for COM-negative binomial distribution. For two independent r.v.'s  $X, Y$ , what is the form of the conditional distribution of  $X$  given  $S = X + Y$ ? Consider the sum of COM-negative binomial r.v.'s with parameters  $(r_x, \nu, p)$  and  $(r_y, \nu, p)$ , then

$$\begin{aligned} P(S = s) &= \sum_{x=0}^s P(X = x) P(Y = s - x) = \sum_{x=0}^s \left( \frac{\Gamma(r_x + x)}{x! \Gamma(r_x)} \right)^\nu \frac{p^x (1 - p)^{r_x}}{C(r_x, \nu, p)} \left( \frac{\Gamma(r_y + s - x)}{(s - x)! \Gamma(r_y)} \right)^\nu \frac{p^{s-x} (1 - p)^{r_y}}{C(r_y, \nu, p)} \\ &= \frac{(1 - p)^{r_x} (1 - p)^{r_y}}{C(r_x, \nu, p) C(r_y, \nu, p)} \sum_{x=0}^s \left( \frac{\Gamma(r_x + x) \Gamma(r_y + s - x)}{x! \Gamma(r_x) (s - x)! \Gamma(r_y)} \right)^\nu p^s \\ &= \frac{(1 - p)^{r_x} (1 - p)^{r_y} [\Gamma(r_x + r_y + s)]^\nu}{C(r_x, \nu, p) C(r_y, \nu, p) [\Gamma(r_x + r_y)]^\nu} \sum_{x=0}^s \left( \binom{s}{x} \frac{B(r_x + x, r_y + s - x)}{B(r_x, r_y)} \right)^\nu p^s. \end{aligned}$$

The conditional distribution  $P(X = k | S = s)$  is

$$\frac{P(X = k)P(Y = s - k)}{P(S = s)} = \left( \binom{s}{k} \frac{B(r_x + k, r_y + s - k)}{B(r_x, r_y)} \right)^\nu / \sum_{x=0}^s \left( \binom{s}{x} \frac{B(r_x + x, r_y + s - x)}{B(r_x, r_y)} \right)^\nu. \quad (16)$$

Using (16), we naturally define the p.m.f. of COM-negative hypergeometric distribution with parameter  $(z, \nu, m, n)$  as follow:

$$P(X = k) = \left( \binom{z}{k} \frac{B(m + k, n + z - k)}{B(m, n)} \right)^\nu / \sum_{x=0}^z \left( \binom{z}{x} \frac{B(m + x, n + z - x)}{B(m, n)} \right)^\nu = \frac{\left( \binom{z}{k} \frac{B(m + k, n + z - k)}{B(m, n)} \right)^\nu}{N(m, n, z, \nu)}, \quad (17)$$

where  $k = 0, 1, 2, \dots, z$  and  $N(m, n, z, \nu)$  is the normalization constant.

When  $\nu = 1$ , COM-negative hypergeometric distribution turns out to be negative hypergeometric distribution, see Wimmer and Altmann (1999), Johnson et al. (2005).

By Patil and Seshadri (1964)'s general characterization theorem for negative binomial, Poisson and Geometric distribution, we know that, given  $X + Y = x + y$ , if the conditional distribution  $X | X + Y$  is negative hypergeometric with parameters  $m$  and  $n$  for all values of the sum  $x + y$ ,

$$P(X = x | X + Y = x + y) = \binom{x + y}{x} \frac{B(m + x, n + y)}{B(m, n)},$$

then  $X$  and  $Y$  both are negative binomial distribution, with parameters  $(m, p)$  and  $(n, p)$ ,

$$f(x) = \frac{\Gamma(m + x)}{x! \Gamma(m)} p^x (1 - p)^m, \quad g(y) = \frac{\Gamma(n + y)}{y! \Gamma(n)} p^y (1 - p)^n$$

respectively (see also Kagan et al. (1973)).

**Lemma 3.1.** (Patil and Seshadri (1964)) Let  $X$  and  $Y$  be independent and both discrete (or both continuous) r.v.'s and suppose  $P(X | X + Y)$  is the function  $c(x, x + y)$ . If  $\frac{c(x + y, x + y)c(0, y)}{c(x, x + y)c(y, y)}$  is of the form  $h(x + y)/h(x)h(y)$  where  $h(\cdot)$  is an arbitrary non-negative function, then

$$f(x) = f(0)h(x)e^{ax}, \quad g(y) = g(0)\frac{h(y)c(0, y)}{c(y, y)}e^{ay}, \quad (18)$$

where  $P(X = x) = f(x) > 0$ ,  $P(Y = y) = g(y) > 0$  and  $f(0), g(0)$  are the corresponding normalizer for  $f(x)$  and  $g(y)$  which make them p.m.f..

Now we apply **Lemma 3.1** for characterizing COM-negative binomial distribution.

**Theorem 3.1.** Let  $X, Y$  be the independent discrete r.v. with  $P(X = x) = f(x) > 0$  and  $P(Y = y) = g(y) > 0$ . If the  $P(X = x | X + Y = x + y)$  is the COM-negative hypergeometric distribution (17) with parameters  $(z = x + y, \nu, m, n)$  for all  $x + y$ , then both  $X$  and  $Y$  have the COM-negative binomial distributions with the parameters  $(m, \nu, p)$  and  $(n, \nu, p)$  respectively.

*Proof.* Note that  $c(x, x + y) = \left( \binom{x + y}{x} \frac{B(m + x, n + x + y - x)}{B(m, n)} \right)^\nu / N(m, n, z, \nu)$ . Then

$$c(a, b) = \left( \binom{b}{a} \frac{B(m + a, n + b - a)}{B(m, n)} \right)^\nu / N(m, n, b, \nu),$$

so

$$\begin{aligned} \frac{c(x + y, x + y)c(0, y)}{c(x, x + y)c(y, y)} &= \left( B(m + x + y, n) \cdot B(m, n + y) / \binom{x + y}{x} B(m + x, n + y) B(m + y, n) \right)^\nu \\ &= \left( \frac{\Gamma(m + x + y)}{(x + y)! \Gamma(m)} / \frac{\Gamma(m + x)}{x! \Gamma(m)} \frac{\Gamma(m + y)}{y! \Gamma(m)} \right)^\nu. \end{aligned}$$

We have  $h(x) = \left(\frac{\Gamma(m+x)}{(x)!\Gamma(m)}\right)^v$ ,  $h(y) = \left(\frac{\Gamma(m+y)}{(y)!\Gamma(m)}\right)^v$ . Apply (18),  $\frac{h(y)c(0,y)}{c(y,y)} = \left(\frac{\Gamma(n+y)}{\Gamma(n)y!}\right)^v$ , let  $p = e^a$  and compare with expression (3), hence the proof is complete.  $\square$

Rao and Rubin (1964) study the following characterization of the Poisson distribution: If  $X$  is a discrete r.v. taking only nonnegative integer values and the conditional distribution of  $Y$  given  $X = x$  is binomial distribution with parameters  $\text{Bi}(x, p)$  ( $p$  does not depend on  $x$ ), then  $X$  follows the Poisson distribution iff

$$P[Y = k] = P[Y = k|Y = X].$$

Based on the COM-negative hypergeometric distribution above and an extension of Rao and Rubin (1964)'s characterization which established by Shanbhag (1977), the Rao-Rubin characterization for COM-negative binomial is obtained.

**Lemma 3.2.** (Shanbhag (1977)) Let  $X, Y$  be the non-negative r.v.'s such that  $P(X = z) = p_z$  with  $p_0 < 1$ ,  $p_z > 0$ , and

$$P(Y = k|X = z) = \frac{a_r b_{z-k}}{\sum_{s=0}^z a_s b_{z-s}} = \frac{a_k b_{z-k}}{c_z}, (k = 0, 1, \dots, z),$$

where  $a_z > 0$  for all  $z \geq 0$ ,  $b_0, b_1 > 0$  and  $b_z \geq 0$  for  $z \geq 2$ , then

$$P(Y = k) = P(Y = k|X = Y), (r = 0, 1, \dots) \quad \text{iff} \quad \frac{p_z}{c_z} = \frac{p_0}{c_0} \theta^z, (z = 0, 1, \dots) \quad \text{for some } \theta > 0.$$

Next, we give the following COM-negative binomial extension of Rao-Rubin characterization. The Shanbhag's result is vital to prove this extension immediately.

**Theorem 3.2.** Let  $X, Y$  be the non-negative integer-valued r.v.'s such that  $P(X = z) = P_z$  with  $P_0 < 1$ ,  $P_z > 0$ , and

$$P(Y = k|X = z) = \left( \binom{z}{k} \frac{B(m+k, n+z-k)}{B(m, n)} \right)^\nu / N(m, n, z, \nu),$$

then  $P(Y = k) = P(Y = k|X = Y), (r = 0, 1, \dots)$  iff  $X$  follows the COM-negative binomial distribution with the parameters  $(m+n, \nu, \theta)$  for some  $\theta > 0$ .

*Proof.* Since the alternative form of negative hypergeometric distribution can be written as

$$\binom{z}{k} \frac{B(m+k, n+z-k)}{B(m, n)} = \frac{\Gamma(m+k)}{B(m, n)k!} \frac{\Gamma(n+z-k)}{B(m, n)(z-k)!} / \frac{\Gamma(m+n+z)}{B(m, n)z!}.$$

From the normalizer in (17), let  $L(m, n, z, \nu) = \frac{N(m, n, z, \nu)}{[B(m, n)]^{-\nu}}$ , we get

$$P(Y = k|X = z) = \frac{[\Gamma(m+k)]^\nu}{(k!)^\nu \sqrt{L(m, n, z, \nu)}} \cdot \frac{[\Gamma(n+z-k)]^\nu}{((z-k)!)^\nu \sqrt{L(m, n, z, \nu)}} / \frac{[\Gamma(m+n+z)]^\nu}{(z!)^\nu} =: \frac{a_k b_{z-k}}{c_z},$$

where  $a_z, b_z, c_z$  satisfy the conditions in **Lemma 3.2**.

Then,  $c_z = \frac{[\Gamma(m+n+z)]^\nu}{(z!)^\nu}$ , comparing with form (3), we conclude that  $X$  is the COM-negative binomial distributed with the parameters  $(m+n, \nu, \theta)$ .  $\square$

### 3.3. COM-negative hypergeometric approximate to COM-negative binomial

We have shown that the conditional distribution of COM-negative binomial r.v.  $X$  given the sum of two COM-negative binomial random variables  $X$  and  $Y$  follows the COM-negative hypergeometric distribution (17). Next, we show that the COM-negative hypergeometric distribution (17) converges to the COM-negative binomial distribution (3).

**Theorem 3.3.** Let  $X$  be the COM-negative hypergeometric r.v. with parameters  $(z, \nu, m, n)$  and p.m.f.  $P(X = k|z, \nu, m, n)$  given by (17), Assume  $m < \infty$ ,  $\frac{z}{n+z} = p^{\frac{1}{\nu}}$ ,  $p \in (0, 1)$ , then

$$\lim_{n \rightarrow \infty} P(X = k|z, \nu, m, n) = \left( \frac{\Gamma(m+k)}{k!\Gamma(m)} \right)^{\nu} p^k (1-p)^m \frac{1}{C(m, \nu, p)}.$$

*Proof.* First, we multiply the p.m.f of COM-negative hypergeometric distribution by the normalization constant (see (17)):

$$\begin{aligned} N(m, n, z, \nu) P(X = k|z, \nu, m, n) &= \left( \frac{z!}{k!(z-k)!} \cdot \frac{\Gamma(m+k)\Gamma(n+z-k)}{\Gamma(m+n+z)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \right)^{\nu} \\ &= \left( \frac{\Gamma(m+k)}{k!\Gamma(m)} \right)^{\nu} \left( \frac{z(z-1)\cdots(z-k+1)\Gamma(n+z-k)\Gamma(n+m)}{(m+n+z-1)(m+n+z-2)\cdots(m+n+z-k)\Gamma(m+n+z-k)\Gamma(n)} \right)^{\nu}. \end{aligned}$$

Using the Stirling's formula for the Gamma function,  $\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(\frac{1}{z}))$ , it yields

$$\begin{aligned} &\frac{\Gamma(n+z-k)}{\Gamma(m+n+z-k)} \cdot \frac{\Gamma(m+n)}{\Gamma(n)} \\ &= \frac{\sqrt{\frac{2\pi}{n+z-k}} \left(\frac{n+z-k}{e}\right)^{n+z-k} \sqrt{\frac{2\pi}{n+m}} \left(\frac{n+m}{e}\right)^{n+m} \left(1 + O\left(\frac{1}{n+z-k}\right)\right) \left(1 + O\left(\frac{1}{n+m}\right)\right)}{\sqrt{\frac{2\pi}{n}} \left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{m+n+z-k}} \left(\frac{m+n+z-k}{e}\right)^{m+n+z-k} \left(1 + O\left(\frac{1}{m+n+z-k}\right)\right) \left(1 + O\left(\frac{1}{n}\right)\right)} \\ &= \sqrt{\frac{n(n+m+z-k)}{(n+z-k)(m+n)}} \left(\frac{n+m}{m+n+z-k}\right)^m \frac{\left(1 + \frac{m}{n}\right)^n}{\left(1 + \frac{m}{n+z-k}\right)^{n+z-k}} \frac{\left(1 + O\left(\frac{1}{n+z-k}\right) + O\left(\frac{1}{n+m}\right)\right)}{\left(1 + O\left(\frac{1}{m+n+z-k}\right) + O\left(\frac{1}{n}\right)\right)}. \end{aligned}$$

From the conditions, we have  $\frac{m+n}{m+n+z-k} \rightarrow 1 - p^{\frac{1}{\nu}}$  as  $n \rightarrow \infty$ . Then

$$\lim_{n \rightarrow \infty} \left( \frac{\Gamma(n+z-k)}{\Gamma(m+n+z-k)} \cdot \frac{\Gamma(m+n)}{\Gamma(n)} \right)^{\nu} = \left(1 - p^{1/\nu}\right)^{m\nu} = (1-p)^m \left( \frac{(1-p^{1/\nu})^{\nu}}{1-p} \right)^m,$$

and

$$\lim_{n \rightarrow \infty} \left( \frac{z(z-1)\cdots(z-k+1)}{(m+n+z-1)(m+n+z-2)\cdots(m+n+z-k)} \right)^{\nu} = p^k.$$

Finally, we obtain

$$\lim_{n \rightarrow \infty} P(X = k|z, \nu, m, n) = \left( \frac{\Gamma(m+k)}{k!\Gamma(m)} \right)^{\nu} p^k (1-p)^m \cdot \lim_{n \rightarrow \infty} [N(m, n, z, \nu)]^{-1} \left( \frac{(1-p^{1/\nu})^{\nu}}{1-p} \right)^m$$

Note that the normalizing constant  $\lim_{n \rightarrow \infty} [N(m, n, z, \nu)]^{-1} \left( \frac{(1-p^{1/\nu})^{\nu}}{1-p} \right)^m$  exists and does not depend on  $k$ . Therefore,  $\lim_{n \rightarrow \infty} P(X = k|z, \nu, m, n)$  is the p.m.f. of CMNB( $m, \nu, p$ ) via comparing with (3).  $\square$

Borges et al. (2014) showed that the COM-Poisson distribution is the limiting distribution of the COM-binomial distribution. Let  $X_m \sim \text{CMB}(m, p, \nu)$  (see (9)) and  $\lambda = m^{\nu} p$  for  $\nu \geq 0$ , then

$$\lim_{m \rightarrow \infty} \frac{\binom{m}{k}^{\nu} p^k (1-p)^{m-k}}{N(m, p, \nu)} = \frac{\lambda^k}{(k!)^{\nu}} \cdot Z^{-1}(\lambda, \nu)$$

namely  $\lim_{m \rightarrow \infty} X_m \sim \text{CMP}(\lambda, \nu)$ .

To sum up, judging from the above mentioned theorems of limiting distribution, we may naturally draw the relationships among some COM-type distributions in Figure 2.

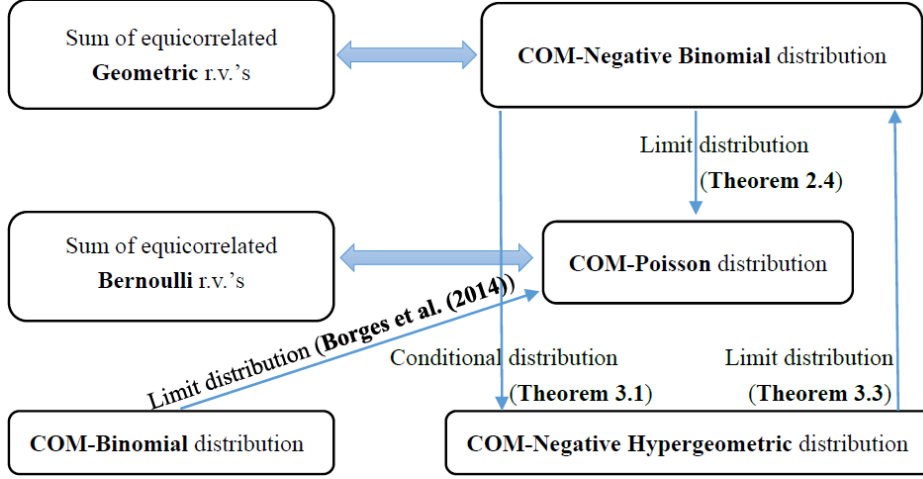


Figure 2: Relationships among some COM-type distributions

#### 3.4. Functional operator characterization

Using recursive formula (6) for COM-negative binomial distribution, an extension of functional operator characterization which arises from the negative binomial approximation literatures by Stein-Chen method is presented. The following lemma also can be derived from the work of Brown and Xia (2001) who give a large class of approximation distribution which is the equilibrium distribution of a birth-death process with arbitrary arrival rate and service rate.

The functional operator characterization is well-known in the Stein-Chen method literatures. **Lemma 3.3** extends the Lemma 1 in Brown and Phillips (1999) for negative binomial approximation.

**Lemma 3.3.** *The r.v.  $W$  has distribution  $\text{CMNB}(r, \nu, p)$  if and only if the equation*

$$\mathbb{E}[W^\nu g(W) - p(W+r)^\nu g(W+1)] = 0. \quad (19)$$

*holds for any bounded function  $g : \mathbb{Z}^+ \rightarrow \mathbb{R}$ .*

*Proof.* Sufficiency: When  $W$  is COM-negative binomial distributed in the form of (3),  $f$  is a bounded function. Then we have

$$\begin{aligned} \mathbb{E}[W^\nu g(W)] &= \sum_{k=0}^{\infty} k^\nu g(k) \left( \frac{\Gamma(k+r)}{k! \Gamma(r)} \right)^\nu \frac{p^k (1-p)^r}{C(r, \nu, p)} = \sum_{k=1}^{\infty} p(k-1+r)^\nu g(k) \left( \frac{\Gamma(k-1+r)}{(k-1)! \Gamma(r)} \right)^\nu \frac{p^{k-1} (1-p)^r}{C(r, \nu, p)} \\ &= \mathbb{E}[p(W+r)^\nu g(W+1)] \end{aligned}$$

Hence  $\mathbb{E}g(W) = 0$ .

Necessity:  $\mathbb{E}g(W) = 0$  for all bounded functions. Letting  $f(W) = 1_{(W=k)}(W)$ , a simple computation shows the recursive formula (6). This verifies that  $W \sim \text{CMNB}(r, \nu, p)$ .  $\square$

#### 4. Generating CMNB-distributed random variables

The inverse transform sampling is a basic method in pseudo-random number sampling, namely for generating sample numbers at random from any probability distribution given its cumulative distribution function. The following lemmas are the basis of inverse method.

**Lemma 4.1.** Suppose  $F(x)$  is the cumulative distribution function of random variable  $\xi$ , then  $\theta = F(\xi)$  is uniformly distributed in interval  $[0, 1]$ .

**Lemma 4.2.** Suppose  $\theta \sim U[0, 1]$ , for any cumulative distribution function  $F(x)$ , let  $\xi = F^{-1}(\theta)$ , then  $\xi \sim F(x)$ , that is,  $F(x)$  is the distribution function of random variable  $\xi$ .

The proof of lemma 4.1 and 4.2 is trivial and can be found in text books of computational statistics. Consider generating random sample from CMNB( $r, \nu, p$ ), since the p.m.f. is

$$P(X = k) = \frac{\left(\frac{\Gamma(r+k)}{k!\Gamma(r)}\right)^\nu p^k (1-p)^r}{\sum_{i=0}^{\infty} \left(\frac{\Gamma(r+i)}{i!\Gamma(r)}\right)^\nu p^i (1-p)^r} = \left(\frac{\Gamma(r+k)}{k!\Gamma(r)}\right)^\nu p^k (1-p)^r \frac{1}{C(r, \nu, p)}, \quad (k = 0, 1, 2, \dots). \quad (20)$$

It follows easily that

$$\begin{cases} P(X = 0) = (1-p)^r \frac{1}{C(r, \nu, p)} \\ \frac{P(X = k)}{P(X = k-1)} = p \cdot \left(\frac{k+r-1}{k}\right)^\nu, \quad (k = 1, 2, \dots). \end{cases} \quad (21)$$

Multiplying these equations yields

$$\begin{aligned} P(X = k) &= p \cdot \left(\frac{k+r-1}{k}\right)^\nu \cdot P(X = k-1) \\ &= p^2 \cdot \left(\frac{k+r-1}{k} \frac{k+r-2}{k-1}\right)^\nu \cdot P(X = k-2) \\ &= p^k \cdot \left(\frac{k+r-1}{k} \frac{k+r-2}{k-1} \dots \frac{r}{1}\right)^\nu \cdot P(X = 0). \end{aligned}$$

Thus, for any non-negative  $k$ , the distribution function  $F(k) = \sum_{i=0}^k P(X = i)$  can be easily calculated. By **Lemma 4.2**, the inverse method alternates between the following steps:

**Step 1:** Let  $j = 0$ ,  $P(X = j) = (1-p)^r / C(r, \nu, p)$ ;

**Step 2:** Generate a random sample  $\theta \sim U[0, 1]$ , and fix its value;

**Step 3:** Let  $F(j+1) = F(j) + P(X = j+1)$ ;

**Step 4:** If  $F(j) < \theta \leq F(j+1)$ , then  $x = j+1$ ; Otherwise, let  $j = j+1$ , and back to Step 3;

We conclude that,  $x$  is the random number drawn from the COM-negative binomial distribution with parameters  $(r, \nu, p)$ .

## 5. Estimations

### 5.1. Estimation by three recursive formulas

The method of recursive formula estimation which give crude estimations of the estimated parameters, is originated with Shmueli et al. (2005), and extended by Imoto (2014). However, this crude estimations can be put into the maximum likelihood estimation (MLE) with Newton-Raphson algorithm as initial values.

First, we note that the p.m.f. of COM-negative binomial has the following recursive formula:

$$\frac{P_{k+1}}{P_k} = p \cdot \left(\frac{k+r}{k+1}\right)^\nu, \quad (k = 0, 1, \dots). \quad (22)$$

Second, applied with the expression of log-concave(log-convex), we have

$$\log\left(\frac{P_k P_{k+2}}{P_{k+1}^2}\right) = \nu \log\left(\frac{k+r+1}{k+2} \cdot \frac{k+1}{r+k}\right). \quad (23)$$

Replace “ $k$ ” by “ $k + 1$ ” in (23). Then we obtain

$$\frac{\log(\frac{P_k P_{k+2}}{P_{k+1}^2})}{\log(\frac{P_{k+1} P_{k+3}}{P_{k+2}^2})} = \frac{\log(\frac{k+r+1}{k+2} \cdot \frac{k+1}{r+k})}{\log(\frac{k+r+2}{k+3} \cdot \frac{k+2}{r+k+1})}. \quad (24)$$

Thus,  $(\nu, r, p)$  could be sequentially solved by the system of equations (22), (23) and (24), if we have the sample p.m.f.

### 5.2. Maximum likelihood estimation

Let r.v.  $X$  be distributed as the COM-negative binomial distribution with parameters  $\theta = (r, \nu, p)$ . We consider the MLE in the case parameters  $r, \nu, p$  are unknown. The likelihood function is

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n P(X = x_i) = \prod_{i=1}^n \left( \frac{\Gamma(r + x_i)}{x_i!} \right)^\nu p^{x_i} \Gamma(r)^{-\nu} (1-p)^r C(r, \nu, p)^{-1} \\ &= \left[ \prod_{i=1}^n \frac{\Gamma(r + x_i)}{x_i!} \right]^\nu p^{\sum_{i=1}^n x_i} \Gamma(r)^{-n\nu} (1-p)^{nr} C(r, \nu, p)^{-n} \end{aligned} \quad (25)$$

where  $n$  is the sample size,  $x_1, x_2, \dots, x_n$  are the observed values, and the log-likelihood function is given by

$$\log L(\theta) = \nu \sum_{i=1}^n \log \frac{\Gamma(r + x_i)}{x_i!} + \log(p) \sum_{i=1}^n x_i + nr \log(1-p) - n\nu \log(\Gamma(r)) - n \log(C(r, \nu, p)) \quad (26)$$

To find the maximum point, for  $\log L(\theta)$ , we take the partial derivatives with respect to  $r, \nu$  and  $p$  and set them equal to zero, hence the likelihood equation is given by

$$\begin{cases} F_1(\theta) = \frac{\partial \log L(\theta)}{\partial r} = \nu \sum_{i=1}^n \psi(r + x_i) + n \log(1-p) - n\nu \psi(r) - n \frac{\partial \log(C(r, \nu, p))}{\partial r} = 0 \\ F_2(\theta) = \frac{\partial \log L(\theta)}{\partial \nu} = \sum_{i=1}^n \log \frac{\Gamma(r + x_i)}{x_i!} - n \log(\Gamma(r)) - n \frac{\partial \log(C(r, \nu, p))}{\partial \nu} = 0 \\ F_3(\theta) = \frac{\partial \log L(\theta)}{\partial p} = \frac{\sum_{i=1}^n x_i}{p} + \frac{nr}{p-1} - n \frac{\partial \log(C(r, \nu, p))}{\partial p} = 0 \end{cases} \quad (27)$$

where  $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$  is called the digamma function. The analytical solutions of the above likelihood equations are not tractable, therefore, numerical optimization method is used to obtain the maximum likelihood estimates. Let  $F(\theta) = (F_1(\theta), F_2(\theta), F_3(\theta))^T$ , then the Fisher information matrix  $I(r, \nu, p)$  would be the Jacobin matrix of  $F(\theta)$ . Given trial values  $\theta_k = (r_k, \nu_k, p_k)$ , applied with the scoring method for solving (27), we can update to  $\theta_{k+1} = (r_{k+1}, \nu_{k+1}, p_{k+1})$  as

$$\begin{pmatrix} r_{k+1} \\ \nu_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} r_k \\ \nu_k \\ p_k \end{pmatrix} + [I(r, \nu, p)]^{-1} F(r_k, \nu_k, p_k) \quad (28)$$

For the initial point  $\theta_0 = (r_0, \nu_0, p_0)$ , we can choose the crude estimated parameters introduced in Subsection 5.1.

### 5.3. Kolmogorov-Smirnov and Chi-squared goodness-of-fit test

In goodness-of-fit test of discrete distributions, Pearson's chi-squared test is a popular choice to check distribution model, which is usually better than the others. The larger p-value, the better goodness-of-fit



would arise from the assumed models. However, when the data come from an assumed model, with  $r$  unknown parameters  $\theta = (\theta_1, \dots, \theta_r)$  and the null hypothesis  $H_0$ , what we are interested in is

$$H_0 : F(x) = F_0(x; \theta_1, \dots, \theta_r),$$

The  $\chi^2$  statistic has the following limiting distribution:

$$\eta = \sum_{i=1}^m \frac{(n_i - n\hat{p}_{0i})^2}{n\hat{p}_{0i}} \xrightarrow{d} \chi^2(m-1-r)$$

where  $n_i$  is the number of class  $i$ , and  $m$  classes  $A_i (i = 1, 2, \dots, m)$  which are the sub-division of  $n$  samples, it satisfies that:

$$\sum_{i=1}^m n_i = n, (-\infty, \infty) = \cup_{i=1}^m A_i, \hat{p}_{i0} =: \frac{\#\{x \in A_i\}}{n}.$$

We also have the another handy expression for calculating  $\chi^2$  statistic:  $\eta = \sum_{i=1}^m \frac{n_i^2}{n\hat{p}_{0i}} - n$ .

One limitation of  $\chi^2$  statistic is that, if degree of freedom are too small, the approximation to the  $\chi^2$  distribution would fail. For example,  $k = m-1-r = 0, 1, 2$ . Since  $\chi^2$  test of small degree of freedom (denotes it by  $k$ ) did not have enough power (namely  $P_\theta(\eta > \eta_0)$  is not large enough). That is to say, for a  $\chi^2$  r.v.  $\eta$ , the p-value is defined by

$$p(\eta_0) = P_\theta(\eta > \eta_0) = \int_{\eta_0}^{+\infty} \frac{1}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} x^{k-1} e^{-\frac{x}{2}} dx$$

tends to small when  $k$  varies from 3 to 1. An attempt at this has been made in the figure below.

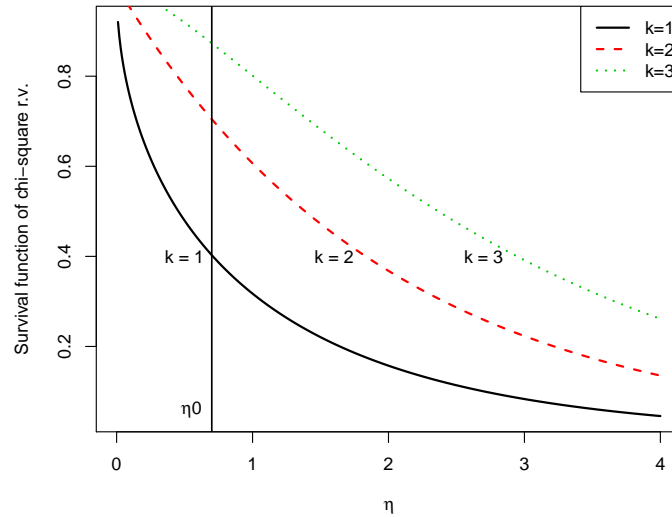


Figure 3: The survival function of  $\chi^2$  r.v. as  $k = 1, 2, 3$

Another drawback is that, with small number of class  $m$ , Haberman (1988) discussed that: “chi-squared test statistics may be asymptotically inconsistent even in cases in which a satisfactory chi-squared approximation exists for the distribution under the null hypothesis.”

Consequently, a better way of comparing distributions is to use a non-parametric test not depending on the parametric assumption of the models. One of notable non-parametric test is Kolmogorov-Smirnov test by using following statistic:

$$D_n = \sup_x \left| \hat{F}_n(x) - F_0(x) \right|$$

which is a nonparametric test of continuous distributions. And it could be modified for discrete distribution:

$$D_n = \sup_x \left| \hat{F}_n(x) - F_0(x) \right| = \max_{i \in I} [\max(|\hat{F}_n(x_i) - F_0(x_i)|, \lim_{x \rightarrow x_i} |\hat{F}_n(x_i) - F_0(x_{i-1})|)],$$

where  $x_i$ 's are the discontinuity points ( $I$  is the countable index set).

Fortunately, Arnold and Emerson (2011) developed the R package “dgof”, which can calculate Kolmogorov-Smirnov test of the discrete distribution. The non-parametric test avoids the assumption of parametric model, hence it is an effective way to evaluate different distributions' performance.

## 6. Applications with simulated and real data

In this section, we will describe four examples of fitting data by COM-negative binomial distribution, and we will compare them with those by the negative binomial and COM-Poisson distribution. The notations, CMNB, NB and CMP distribution, in the following tables, are representing the COM-negative binomial, negative binomial and COM-Poisson, respectively.

By using scoring method in subsection 5.2, we could estimated the CMNB distribution with three parameters. The original and expected frequencies, parameter estimators (obtained by maximum likelihood method), the  $\chi^2$  and K-S statistics, and corresponding  $p$ -values. are all being considered in the tables below.

### 6.1. Simulated Data

In this subsection, the inverse method mentioned in section 4 were used to generate random variables from CMNB, NB and COM-Poisson distribution with certain parameters. And we compare the performance of the fit by CMNB, NB and COM-Poisson distribution.

**Example 1.** As our first simulation example, we consider the original data that are following the NB distribution. We use inverse method to draw 10000 random samples from NB distribution with parameter ( $r = 1, p = 0.5$ ), and fit the data with above-mentioned distributions, the results are shown in the following table 1.

Sample values	Frequency	Fitted Values		
		CMNB	NB	CMP
0	5060	5054	5049	5019
1	2480	2494	2492	2502
2	1199	1237	1236	1248
3	638	615	614	622
4	318	306	306	310
5	165	152	152	155
6	74	76	76	77
7	33	38	38	38
8	20	19	19	19
9	8	9	9	10
10	4	5	5	5
11	1	2	2	2
Total	10000	10007	9998	10007
par1		0.97	0.99	0.50
par2		1.00	0.50	0.00
par3		0.51		
$\chi^2$		5.29	5.44	5.83
d.f. of $\chi^2$		8.00	9.00	9.00
p value of $\chi^2$		0.73	0.79	0.76
K-S		0.002199	0.003976	0.004451
p value of K-S		1.000000	0.997435	0.988831

In this case, although the data are generated from NB distribution, the CMNB distribution can also be recognized as the true distribution, as the estimator of  $\nu$  is 1. That is to say, the data can be seen negative binomial distributed, and note that the result of  $\chi^2$  test and K-S test is nearly the same, all of these indicate that the CMNB distribution is an extension of NB distribution.

**Example 2.** In this example, we will evaluate the performance of NB distribution, when the original data are generated from CMNB distribution. Assume  $X_1, X_2, \dots, X_n$  are independent and identically CMNB distributed random variables with parameters  $(r, \nu, p)$ . Let  $n = 10000$  and  $(r, \nu, p) = (0.005, 0.1, 0.5)$ , we can generate  $n$  sample points  $x_1, x_2, \dots, x_n$  through inverse method introduced in section 4. The fitting results are summarized below:

Table 2: Fitting results of simulated data from CMNB(0.005, 0.1 0.5)

Sample values	Frequency	Fitted Values		
		CMNB	NB	CMP
0	6442	6449	6421	5937
1	1866	1878	1951	2413
2	874	869	837	981
3	435	415	394	399
4	188	201	194	162
5	101	98	98	66
6	55	48	50	27
7	19	24	26	11
8	7	12	14	4
9	8	6	7	2
10	2	3	4	1
11	2	1	2	0
12	1	1	1	0
Total	10000	10005	9999	10003
par1		0.01	0.55	0.41
par2		0.12	0.45	0.00
par3		0.50		0.44
$\chi^2$		7.26	16.38	281.94
d.f. of $\chi^2$		9.00	10.00	10.00
p value of $\chi^2$		0.61	0.09	0.00
K-S		0.001484	0.006484	0.050678
p value of K-S		1.000000	0.794579	0.000000

From table 2, we note first that in  $\chi^2$  test, the CMNB distribution's  $\chi^2$  statistic 7.26 is dramatically smaller than NB distribution's 16.38, while the corresponding p-values are 0.61 and 0.09 respectively, which implies that NB distribution may not be a reasonable fit. As for the result of the K-S test, it suggests that CMNB distribution performs best, because the p-value of CMNB distribution is approximately 1, while NB's and COM-Poisson's are 0.79 and 0.

## 6.2. Real data analysis

In this subsection, the distributions mentioned above are considered here to analyze real actuarial claim data that have ultrahigh zero-inflated and overdispersion properties, then compare its Kolmogorov-Smirnov test and Chi-squared test.

**Example 3.** Let us consider the claim counts of the third party liability vehicle insurance (see Willmot (1987) for data set in an Zaire insurance company) which correspond to claims from 4000 vehicle policies. Gmez-Dniz (2014) analyzed the data using negative binomial distribution and found that it is a reasonable fit. We analyze the data by CMNB, NB and CMP distribution, and the results are summarized below:

Table 3: Fit of Willmot2 data				
No. of claims	Frequency	Fitted Values		
		CMNB	NB	CMP
0	3719	3720	3719	3681
1	232	231	230	294
2	38	39	40	23
3	7	8	8	2
4	3	2	2	0
5	1	1	0	0
Total	4000	4001	3999	3991
par1		0.57	0.22	0.08
par2		3.06	0.71	0.00
par3		0.35		
$\chi^2$		1.01	1.56	173.28
d.f. of $\chi^2$		2.00	3.00	3.00
p value of $\chi^2$		0.60	0.67	0.00
K-S		0.000250	0.000500	0.009500
p value of K-S		1.000000	1.000000	0.863178

It's a well known fact that NB distribution is a popular choice to fit the claim data in actuarial science. However findings in Table 3 suggest that, CMNB might be a better choice for fitting this data. It appears that, for chi-square statistic, fitting the chi-square statistic with the NB distribution, it turns out to be 1.56, which is dramatically larger than CMNB's 1.01, and the p-values correspond to the  $\chi^2$  statistics is nearly the same. As for Kolmogorov-Smirnov test, although the p-value is both 1, it still can be seen that the K-S statistic of CMNB is slightly smaller than NB's. All of these show that the COM-negative binomial is superior to the fit the data.

**Example 4.** For this example, we use the car insurance claim data of a Chinese insurance company (see Wang and Lei (2000)), which were modeled with negative binomial distribution. Total insurance policies are  $n = 35072$ . We analyze the data with the above-mentioned distributions, and the results are shown in table 4.

Table 4: Fit of Car insurance claim data				
No. of Claims	Frequency	Fitted Values		
		CMNB	NB	CMP
0	27141	27177	27166	26599
1	5789	5666	5664	6430
2	1443	1554	1563	1554
3	457	466	467	376
4	155	146	145	91
5	56	47	46	22
6	27	15	15	5
7	2	5	5	1
8	1	2	2	0
9	1	1	1	0
Total	35072	35079	35074	35078
par1		0.95	0.61	0.24
par2		10.40	0.66	0.00
par3		0.36		
$\chi^2$		24.16	27.88	300.60
d.f. of $\chi^2$		6.00	7.00	7.00
p value of $\chi^2$		0.00	0.00	0.00
K-S		0.002667	0.002905	0.015584
p value of K-S		0.964199	0.928729	0.000000

As can be observed from table 4, CMNB is the clear winner. CMNB distribution outperforms other distributions by either  $\chi^2$  test or K-S test, as  $\chi^2$  statistic and K-S statistic of CMNB is slightly smaller than that of NB's, and dramatically smaller than CMP's. It shows that their performance is on the whole dominated by CMNB distribution.

## 7. Further researches

Many discrete distributions in aspects of related statistical models are widely proposed in plenty of research articles. The COM-negative binomial distribution can be used in driving count data in some generalized linear model. For example, Poisson regression, COM-Poisson regression and negative binomial regression, whose driving distribution of count data are actually the special case or limiting case of COM-negative binomial distribution, see Figure 2 for a visual relations. Even in the popular logistic regression, the assuming distribution lying in this model is Bernoulli distribution which is a limiting case of COM-Poisson distribution. Unless the generalized linear models, the discrete frailty item as a random effect in survival analysis model (or Long-term survival models, see Rodrigues et al. (2012)), which could be considered to set some flexible distributions. Besides the researches of model, in our paper, both conditional distribution and Stein identity characterizations are obtained. Test of statistic based on the novel characterization is more reasonable, since that the distribution-free and omnibus goodness-of-fit test like Kolmogorov-Smirnov and Chi-squared does not depend on certain types(or families) of distributions, may have their own drawbacks. For example, some new tests of a class of count distributions which includes the Poisson is constructed from Stein identity characterization(see Meintanis and Nikitin (2008)), and some tests of logarithmic series distribution comes from conditional distribution characterization(see Ramalingam and Jagbir (1984)).

## 8. Acknowledgments

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## 9. References

### References

- Arnold, T. B., Emerson, J. W. (2011). Nonparametric goodness-of-fit tests for discrete null distributions. *The R Journal*, 3(2), 34-39.
- Borges, P., Rodrigues, J., Balakrishnan, N., Bazn, J. (2014). A COM-Poisson type generalization of the binomial distribution and its properties and applications. *Statistics & Probability Letters*, 87, 158-166.
- Brown, T. C., Phillips, M. J. (1999). Negative binomial approximation with Stein's method. *Methodology and Computing in Applied Probability*, 1(4), 407-421.
- Brown, T. C., Xia, A. (2001). Stein's method and birth-death processes. *Annals of probability*, 29(3), 1373-1403.
- Chakraborty, S., Imoto, T. (2016). Extended Conway-Maxwell-Poisson distribution and its properties and applications. *Journal of Statistical Distributions and Applications*, 3(1), 5.
- Conway, R. W., Maxwell, W. L. (1962). A queuing model with state dependent service rates. *Journal of Industrial Engineering*, 12(2), 132-136.
- Deng, Z., Tan, Z. (2009). Fitting three-parameter Poisson-Tweedie model to claim frequency and test the location parameter. *Mathematical Theory and Applications*, 29(1), 51-55. (in Chinese) [http://en.cnki.com.cn/Article\\_en/CJFDTOTAL-LLYY200901014.htm](http://en.cnki.com.cn/Article_en/CJFDTOTAL-LLYY200901014.htm)
- Denuit, M. (1997). A new distribution of Poisson-type for the number of claims. *Astin Bulletin*, 27(02), 229-242.
- Denuit, M., Maréchal, X., Pitrebois, S., Walhin, J. F. (2007). *Actuarial modelling of claim counts: Risk classification, credibility and bonus-malus systems*. Wiley.
- Gmez-Dniz, E., Sarabia, J. M., Caldern-Ojeda, E. (2011). A new discrete distribution with actuarial applications. *Insurance: Mathematics and Economics*, 48(3), 406-412.
- Gmez-Dniz, E., Caldern-Ojeda, E. (2014). Unconditional distributions obtained from conditional specification models with applications in risk theory. *Scandinavian Actuarial Journal*, 2014(7), 602-619.
- Gupta, R. C., Sim, S. Z., Ong, S. H. (2014). Analysis of discrete data by Conway-Maxwell Poisson distribution. *ASTA Advances in Statistical Analysis*, 1-17.
- Haberman, S. J. (1988). A warning on the use of chi-squared statistics with frequency tables with small expected cell counts. *Journal of the American Statistical Association*, 83(402), 555-560.
- Ibragimov, I. D. A. (1956). On the composition of unimodal distributions. *Theory of Probability & Its Applications*, 1(2), 255-260.
- Imoto, T. (2014). A generalized Conway-Maxwell-Poisson distribution which includes the negative binomial distribution. *Applied Mathematics and Computation*, 247, 824-834.
- Johnson, N. L., Kemp, A. W., Kotz S. (2005). *Univariate discrete distributions*, 3rd. Wiley.

- Kadane, J. B. (2016). Sums of possibly associated Bernoulli variables: The Conway-Maxwell-Binomial distribution. *Bayesian Analysis*, 11(2), 403-420.
- Kagan, A. M., Linnik, Y. V., Rao, C. R. (1973). *Characterization problems in mathematical statistics*, Wiley.
- Kaas, R., Denuit, M., Dhaene, J., Goovaerts, M. (2008). *Modern Actuarial Risk Theory Using R*, 2ed. Springer, Berlin.
- Kokonendji, C. C., Mizere, D., Balakrishnan, N. (2008). Connections of the Poisson weight function to overdispersion and underdispersion. *Journal of Statistical Planning and Inference*, 138(5), 1287-1296.
- Meintanis, S. G., Nikitin, Y. Y. (2008). A class of count models and a new consistent test for the Poisson distribution. *Journal of Statistical Planning and Inference*, 138(12), 3722-3732.
- Patil, G. P., Seshadri, V. (1964). Characterization theorems for some univariate probability distributions. *Journal of the Royal Statistical Society. Series B (Methodological)*, 286-292.
- Rao, C. R., Rubin, H. (1964). On a characterization of the Poisson distribution. *Sankhyā: The Indian Journal of Statistics, Series A*, 295-298.
- Rényi, A. (1961). On measures of entropy and information. In *Fourth Berkeley symposium on mathematical statistics and probability* (Vol. 1, pp. 547-561).
- Ramalingam, S., Jagbir, S. (1984). A characterization of the logarithmic series distribution and its application. *Communications in Statistics-Theory and Methods*, 13(7), 865-875.
- Rodrigues, J., de Castro, M., Cancho, V. G., Balakrishnan, N. (2009). COM-Poisson cure rate survival models and an application to a cutaneous melanoma data. *Journal of Statistical Planning and Inference*, 139(10), 3605-3611.
- Sellers, K. F., Borle, S., Shmueli, G. (2012). The COM-Poisson model for count data: a survey of methods and applications. *Applied Stochastic Models in Business and Industry*, 28(2), 104-116.
- Shaked, M., Shanthikumar, J. G. (2007). *Stochastic orders*. Springer.
- Shanbhag, D. N. (1977). An extension of the Rao-Rubin characterization of the Poisson distribution. *Journal of Applied Probability*, 640-646.
- Shmueli, G., Minka, T. P., Kadane, J. B., Borle, S., Boatwright, P. (2005). A useful distribution for fitting discrete data: revival of the Conway-Maxwell-Poisson distribution. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 54(1), 127-142.
- Steutel, F. W. (1970). Preservation of infinite divisibility under mixing and related topics. *MC Tracts*, 33, 1-99.
- Steutel, F. W., Van Harn, K. (2003). *Infinite divisibility of probability distributions on the real line*. CRC Press, New York.
- Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *Journal of statistical physics*, 52(1-2), 479-487.
- Temme, N. M. (2011). *Special functions: An introduction to the classical functions of mathematical physics*. Wiley.
- Wang, L.M., Lei, Y.L., (2000). Simulation and EM algorithm for the distribution of number of claim in the heterogeneous portfolio. *Communications on Applied Mathematics and Computational Science*. 14(2) pp. 71-78.
- Willmot, G. E. (1987). The Poisson-inverse Gaussian distribution as an alternative to the negative binomial. *Scandinavian Actuarial Journal*, 1987(3-4), 113-127.
- Wimmer, G., Köhler, R., Grotjahn, R., Altmann, G. (1994). Towards a theory of word length distribution. *Journal of Quantitative Linguistics*, 1(1), 98-106.
- Wimmer, G., Altmann, G. (1999). *Thesaurus of univariate discrete probability distributions*. Stamm.
- Zhang, H., Liu, Y., Li, B. (2014). Notes on discrete compound Poisson model with applications to risk theory. *Insurance: Mathematics and Economics*, 59, 325-336.
- Zhang, H., Li, B. (2016). Characterizations of discrete compound Poisson distributions. *Communications in Statistics-Theory and Methods*, 45(22), 6789-6802.
- Zhang, H., Li, B., Kerns, G. J. (2017). A characterization of signed discrete infinitely divisible distributions. *arXiv preprint arXiv:1701.03892*. To appear in *Studia Scientiarum Mathematicarum Hungarica*.
- Zygmund, A. (2002). *Trigonometric series*. Cambridge University Press, Cambridge.